

# Logarithmical Blow-up Criteria for the Nematic Liquid Crystal Flows <sup>\*</sup>

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## Abstract

We investigate the blow-up criterion for the local in time classical solution of the nematic liquid crystal flows in dimension two and three. More precisely,  $0 < T_* < +\infty$  is the maximal time interval if and only if (i) for  $n = 3$ ,

$$\int_0^{T_*} \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty,$$

or

$$\int_0^{T_*} \frac{\|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty;$$

and (ii) for  $n = 2$ ,

$$\int_0^{T_*} \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty.$$

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## 1 Introduction

In this paper, we are interested in the following Cauchy problem of the flow of the nematic liquid crystal material in  $n$ -dimensions ( $n = 2$  or  $3$ ):

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d) \quad \text{in } \mathbb{R}^n \times (0, +\infty), \quad (1.1)$$

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$$\partial_t d + (u \cdot \nabla) d = \gamma(\Delta d + |\nabla d|^2 d) \quad \text{in } \mathbb{R}^n \times (0, +\infty), \quad (1.2)$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty), \quad (1.3)$$

$$(u, d)|_{t=0} = (u_0, d_0) \quad \text{in } \mathbb{R}^n, \quad (1.4)$$

where  $u(x, t) : \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{R}^n$  is the unknown velocity field of the flow,  $P(x, t) : \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{R}$  is the scalar pressure and  $d : \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{S}^2$ , the unit sphere in  $\mathbb{R}^3$ , is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow,  $\nabla \cdot u = 0$  represents the incompressible condition,  $u_0$  is a given initial velocity with  $\nabla \cdot u_0 = 0$  in distribution sense,  $d_0 : \mathbb{R}^n \rightarrow \mathbb{S}^2$  is a given initial liquid crystal orientation field, and  $\nu, \lambda, \gamma$  are positive constants. The notation  $\nabla d \odot \nabla d$  denotes the  $n \times n$  matrix whose  $(i, j)$ -th entry is given by  $\partial_i d \cdot \partial_j d$  ( $1 \leq i, j \leq n$ ). Since the concrete values of the constants  $\nu, \lambda$  and  $\gamma$  do not play a special role in our discussion, for simplicity, we assume that they all equal to one throughout this paper.

The system (1.1)–(1.4) is a simplified version of the Ericksen-Leslie model [4, 13], which can be viewed as the incompressible Navier–Stokes equations (the case  $d \equiv 1$ , see [8, 12, 14]) coupling the heat flow of a harmonic map (the case  $u \equiv 0$ , see [3, 14, 26]). Mathematical analysis of the system (1.1)–(1.4) was initially studied by a series of papers by Lin [16] and Lin and Liu [18, 19]. Later on, there are many extensive studies devote to the nematic liquid crystal flows, see [5, 6, 7, 15, 17, 20, 21, 22, 24, 26, 27] and references therein. For instance, when the dimension  $n = 2$ , Lin, Lin and Wang [17] established global existence of Leray-Hopf type weak solutions to (1.1)–(1.4) on bounded domain in  $\mathbb{R}^2$  under suitable initial and boundary value conditions. Li and Wang in [15] established the existence of local strong solution with large initial value and the global strong solution with small initial value for the initial-boundary value problem of system (1.1)–(1.4). Wang in [26] proved that if the initial data  $(u_0, d_0) \in BMO^{-1} \times BMO$  is sufficiently small, then system (1.1)–(1.4) exists a global mild solution. Lin and Wang [20] established that when the initial data  $(u_0, d_0)$  satisfying  $u_0, \nabla d_0 \in L^n(\mathbb{R}^n)$ , the solution  $(u, d) \in C([0, T]; L^n(\mathbb{R}^n)) \times C([0, T]; \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{S}^2))$  to system (1.1)–(1.4) is unique.

In the present paper, we are interesting in the short time classical solution to the system (1.1)–(1.4). Since the strong solutions of the heat flow of harmonic maps must be blowing up at finite time [2], we cannot expect that (1.1)–(1.4) has a global smooth solution with general initial data. It is well-known that if the initial velocity  $u_0 \in H^s(\mathbb{R}^n, \mathbb{R}^n)$  with  $\nabla \cdot u_0$  and  $d_0 \in H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)$  for  $s \geq n$ , then there exists  $0 < T_* < +\infty$  depending only on the initial value such that the system (1.1)–(1.4) has a unique local classical solution  $(u, d) \in \mathbb{R}^n \times [0, T_*)$  satisfying (see for example [27])

$$\begin{aligned} u &\in C([0, T]; H^s(\mathbb{R}^n, \mathbb{R}^n)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^n, \mathbb{R}^n)) \quad \text{and} \\ d &\in C([0, T]; H^{s+1}(\mathbb{R}^n, \mathbb{S}^2)) \cap C^1([0, T]; H^s(\mathbb{R}^n, \mathbb{S}^2)) \end{aligned} \quad (1.5)$$

for all  $0 < T < T_*$ . Here, we emphasize that such an existence theorem gives no indication as to whether solutions actually lose their regularity or the manner in which they may do so. Assume that such  $T_*$  is the maximum value for (1.5) holds, the purpose of this paper is to characterize such a  $T_*$ .

For the well-known Navier-Stokes equations with dimension  $n \geq 3$ , the Serrin conditions (see [23, 14]) state that if  $0 < T_* < \infty$  is the first finite singular time of the smooth solutions  $u$ , then  $u$  does not belong to the class  $L^\alpha(0, T_*; L^\beta(\mathbb{R}^n))$  for all  $\frac{2}{\alpha} + \frac{n}{\beta} \leq 1$ ,  $2 < \alpha < \infty$ ,  $n < \beta < \infty$ . Beale, Kato and Majida in [1] proved that the vorticity  $\omega = \nabla \times u$  does not belong to  $L^1(0, T_*; L^\infty(\mathbb{R}^n))$  if  $T_*$  is the first finite singular time. Later on, Kozono and Taniuchi [12], Kozono, Ogawa and Taniuchi [11] and Guo and Gala [9] improved the results of [1] into BMO and Besov space, more precisely, if  $T_*$  is the first singular time, then there hold

$$\begin{aligned} \int_0^{T_*} \|\omega\|_{BMO} dt &= \infty; \\ \int_0^{T_*} \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0}}{\sqrt{1 + \ln(1 + \|\omega\|_{\dot{B}_{\infty,\infty}^0})}} dx &= \infty, \end{aligned}$$

where  $\dot{B}_{\infty,\infty}^0$  denotes the homogeneous Besov space. On the other hand, as for the heat flow of harmonic maps into  $\mathbb{S}^2$ , Wang [26] established that for  $n \geq 2$ , the condition  $\nabla d \in L^\infty(0, T; L^n(\mathbb{R}^n))$  implies that the solution  $d$  is regular on  $(0, T]$ , i.e.,  $d \in C^\infty((0, T] \times \mathbb{R}^n)$ . For the system (1.1)–(1.4), when dimension  $n = 2$ , Lin, Lin and Wang obtained that the local smooth solution  $(u, d)$  to (1.1)–(1.4) can be continued past any time  $T > 0$  provided that there holds

$$\int_0^T \|\nabla d(\cdot, t)\|_{L^4}^4 dt < \infty.$$

Huang and Wang [7] established that

$$\begin{aligned} \int_0^{T_*} (\|\omega\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2) dx &= \infty \quad \text{when dimension } n = 3; \\ \int_0^{T_*} \|\nabla d\|_{L^\infty}^2 dx &= \infty \quad \text{when dimension } n = 2, \end{aligned}$$

where  $0 < T_* < \infty$  is the first finite singular time. Motivated by the above cited papers, the purpose of this paper is to establish blow-up criteria for local smooth solutions of system (1.1)–(1.4) in term of the homogeneous Besov spaces.

Our main results are as follows:

**Theorem 1.1** *For  $n = 3$ ,  $u_0 \in H^3(\mathbb{R}^3, \mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$  and  $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$ , let  $T_* > 0$  be the maximum value such that the nematic liquid crystal flow (1.1)–(1.4) has a unique solution  $(u, d)$  satisfying (1.5). If  $T_* < +\infty$ , then*

$$\int_0^{T_*} \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty, \quad (1.6)$$

and

$$\int_0^{T_*} \frac{\|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty, \quad (1.7)$$

where  $\omega := \nabla \times u$  is the vorticity. In particular, it holds that

$$\begin{aligned} \limsup_{t \rightarrow T_*} (\|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2) &= 0; \\ \limsup_{t \rightarrow T_*} (\|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}) &= 0. \end{aligned}$$

**Remark 1.2** 1. By the Sobolev imbedding  $L^\infty(\mathbb{R}^3) \subset \dot{B}_{\infty,\infty}^0(\mathbb{R}^3)$ , it is easy to see that the condition (1.6) is an extension that of [7].

2. Notice that  $\nabla u \in \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$  is equivalent to  $u \in \dot{B}_{\infty,\infty}^0$ , it follows that the condition (1.7) can be replaced by the following condition:

$$\int_0^{T_*} \frac{\|u\|_{\dot{B}_{\infty,\infty}^0}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|u\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty.$$

In particular, there holds

$$\limsup_{t \rightarrow T_*} (\|u\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}) = 0.$$

As a byproduct of our proof of Theorem 1.1, we obtain the following corresponding criterion in dimension two. More precisely, we have

**Theorem 1.3** For  $n=2$ ,  $u_0 \in H^2(\mathbb{R}^2, \mathbb{R}^2)$  with  $\nabla \cdot u_0 = 0$  and  $d_0 \in H^3(\mathbb{R}^2, \mathbb{S}^2)$ , let  $T_* > 0$  be the maximum value such that the nematic liquid crystal flow (1.1)–(1.4) has a unique solution  $(u, d)$  satisfying (1.5). If  $T_* < +\infty$ , then

$$\int_0^{T_*} \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt = \infty. \quad (1.8)$$

In particular, there holds

$$\limsup_{t \rightarrow T_*} \|\nabla d\|_{\dot{B}_{\infty,\infty}^0} = 0.$$

The remaining of the paper is written as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we prove Theorem 1.3. Throughout the paper,  $C$  denotes a constant and may change from line to line;  $\|\cdot\|_X$  denotes the norm of space  $X(\mathbb{R}^3)$  or  $X(\mathbb{R}^2)$ .

## 2 The proof of Theorem 1.1

In this Section, we shall give the proof of Theorem 1.1. Before going to the proof, we first review the following two inequalities, the first one can be found in [11] and the second one can be found in [9]:

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{\dot{B}_{\infty,\infty}^0} \ln^{\frac{1}{2}}(1 + \|f\|_{H^{s-1}})) \quad (2.1)$$

for  $f \in H^{s-1}(\mathbb{R}^n)$  with  $s > \frac{n}{2} + 1$  and  $n \geq 2$ .

$$\|f\|_{L^4} \leq \|f\|_{\dot{B}_{\infty,\infty}^{-1}}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}} \quad (2.2)$$

for  $f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$ .

*Case I:* We now give the proof of Theorem 1.1 under the assumption (1.6).

Since we deal with the local smooth solutions, and notice that  $[0, T_*)$  is the maximal existence interval of local smooth solution associated with initial value  $(u_0, d_0)$ . We prove Theorem 1.1 arguing by contradiction. Suppose, that (1.6) is not true. Then there is  $0 < M < \infty$  such that

$$\int_0^{T_*} \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt \leq M. \quad (2.3)$$

We will show that if assumption (2.3) holds, then there holds

$$\lim_{t \rightarrow T_*^-} (\|\nabla^3 u(t, \cdot)\|_{L^2}^2 + \|\nabla^4 d(t, \cdot)\|_{L^2}^2) \leq C, \quad (2.4)$$

for some positive constant  $C$  depends only on  $u_0, d_0, T_*$  and  $M$ . The estimate (2.4) is enough to extend the smooth solution  $(u, d)$  beyond to  $T_*$ . That is to say,  $[0, T_*)$  is not a maximal interval of existence, which leads to the contradiction.

We first taking  $\nabla \times$  on (1.1), it follows that

$$\omega_t - \Delta \omega + u \cdot \nabla \omega = \omega \cdot \nabla u - \nabla \times (\Delta d \cdot \nabla d), \quad (2.5)$$

where we have used the facts that  $\nabla \cdot (\nabla d \otimes \nabla d) = \nabla(\frac{|\nabla d|^2}{2}) + \Delta d \cdot \nabla d$  and  $\nabla \times \nabla(\frac{|\nabla d|^2}{2}) = 0$ . Multiplying (2.5) with  $\omega$  and integrating over  $\mathbb{R}^3$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|\omega(\cdot, t)\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 = \int_{\mathbb{R}^3} [(\omega \cdot \nabla) u \cdot \omega + (\Delta d \cdot \nabla d) \cdot \nabla \times \omega] dx := I_1 + I_2, \quad (2.6)$$

where we have used the fact that  $\operatorname{div} u = 0$  implies that  $\int_{\mathbb{R}^3} (u \cdot \nabla) \omega \cdot \omega dx = \frac{1}{2} \int_{\mathbb{R}^3} (u \cdot \nabla) |\omega|^2 dx = 0$ . By using the Hölder's inequality and (2.1) with  $s = 3$ , we can estimate  $I_1$  as

$$\begin{aligned} I_1 &\leq C \|\omega\|_{L^\infty} \|\nabla u\|_{L^2} \|\omega\|_{L^2} \leq C \|\omega\|_{L^\infty} \|\omega\|_{L^2}^2 \\ &\leq C(1 + \|\omega\|_{\dot{B}_{\infty,\infty}^0} \ln^{\frac{1}{2}}(e + \|\Lambda^2 \omega\|_{L^2})) \|\omega\|_{L^2}^2 \\ &\leq C(1 + \|\omega\|_{\dot{B}_{\infty,\infty}^0} \ln^{\frac{1}{2}}(e + \|\Lambda^3 u\|_{L^2})) \|\omega\|_{L^2}^2 \\ &\leq C \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0}}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} \ln^{\frac{1}{2}}(e + \|\Lambda^3 u\|_{L^2}) \\ &\quad \times (1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}))^{\frac{1}{2}} \|\omega\|_{L^2}^2 + C \|\omega\|_{L^2}^2 \\ &\leq C \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0}}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} \ln^{\frac{1}{2}}(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2}) \\ &\quad \times (1 + \ln(e + \|\Lambda^2 \omega\|_{L^2} + \|\Lambda^3 \nabla d\|_{L^2}))^{\frac{1}{2}} \|\omega\|_{L^2}^2 + C \|\omega\|_{L^2}^2 \end{aligned}$$

$$\leq C \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0}}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} \ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2}) \|\omega\|_{L^2}^2 + C \|\omega\|_{L^2}^2, \quad (2.7)$$

where  $\Lambda := (-\Delta)^{\frac{1}{2}}$ , and we have used the fact

$$\nabla u = (-\Delta)^{-1} \nabla (\nabla \times \omega) \text{ implies } \|\nabla u\|_{L^2} \leq C \|\omega\|_{L^2},$$

and the following standard Sobolev imbedding

$$H^3(\mathbb{R}^3) \subseteq H^2(\mathbb{R}^3) \subseteq L^\infty(\mathbb{R}^3) \subseteq BMO(\mathbb{R}^3) \subseteq \dot{B}_{\infty,\infty}^0(\mathbb{R}^3). \quad (2.8)$$

As for  $I_2$ , by using the Hölder's inequality and (2.1) with  $s = 4$ , we get

$$\begin{aligned} I_2 &\leq C \|\Delta d\|_{L^2} \|\nabla d\|_{L^2} \|\nabla \omega\|_{L^2} \leq C \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} \|\nabla \omega\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + C (1 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2 \ln(e + \|\Lambda^3 \nabla d\|_{L^2})) \|\Delta d\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} \ln(e + \|\Lambda^3 \nabla d\|_{L^2}) \\ &\quad \times (1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}))^{\frac{1}{2}} \|\Delta d\|_{L^2}^2 + C \|\Delta d\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} [\ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2})]^{\frac{3}{2}} \|\Delta d\|_{L^2}^2 \\ &\quad + C \|\Delta d\|_{L^2}^2. \end{aligned} \quad (2.9)$$

Inserting estimates (2.7) and (2.9) into (2.6), it follows that

$$\begin{aligned} \frac{d}{dt} \|\omega(\cdot, t)\|_{L^2}^2 + \frac{3}{2} \|\nabla \omega\|_{L^2}^2 &\leq C (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ &\quad + C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} [\ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2})]^{\frac{3}{2}} (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2). \end{aligned} \quad (2.10)$$

Taking  $\Delta$  on equation (1.3), multiplying  $\Delta d$  and integrating over  $\mathbb{R}^3$ , one obtains

$$\frac{1}{2} \frac{d}{dt} \|\Delta d(\cdot, t)\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 = - \int_{\mathbb{R}^3} \Delta(u \cdot \nabla d) \cdot \Delta d dx + \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d) \cdot \Delta d dx := I_3 + I_4. \quad (2.11)$$

Notice that the fact  $\operatorname{div} u = 0$  implies that

$$\int_{\mathbb{R}^3} (u \cdot \nabla) \Delta d \cdot \Delta d dx = \frac{1}{2} \int_{\mathbb{R}^3} u \cdot \nabla |\Delta d|^2 dx = 0,$$

and the equality

$$\nabla \times \omega = \nabla \times (\nabla \times u) = \nabla (\nabla \cdot u) - \Delta u = -\Delta u \text{ implies } \|\Delta u\|_{L^2} \leq C \|\nabla \omega\|_{L^2}.$$

Hence we can estimate  $I_3$  as

$$\begin{aligned}
I_3 &\leq \int_{\mathbb{R}^3} |\Delta u| |\nabla d| |\Delta d| dx + 2 \int_{\mathbb{R}^3} |\nabla u| |\nabla^2 d| |\Delta d| dx \\
&\leq \|\Delta u\|_{L^2} \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^4} \|\Delta d\|_{L^4} \\
&\leq \|\nabla \omega\|_{L^2} \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} + \|\omega\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla \Delta d\|_{L^2} \\
&\leq \frac{1}{4} (\|\nabla \omega\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) + C \|\nabla d\|_{L^\infty}^2 (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
&\leq \frac{1}{4} (\|\nabla \omega\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) + C(1 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2 \ln(e + \|\Lambda^3 \nabla d\|_{L^2})) (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
&\leq \frac{1}{4} (\|\nabla \omega\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) + C(\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \\
&\quad C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} [\ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2})]^{\frac{3}{2}} (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \quad (2.12)
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequalities

$$\|\nabla^2 d\|_{L^4} \leq \|\nabla d\|_{L^\infty}^{\frac{1}{2}} \|\nabla \Delta d\|_{L^2}^{\frac{1}{2}} \quad \text{and} \quad \|\Delta d\|_{L^4} \leq \|\nabla d\|_{L^\infty}^{\frac{1}{2}} \|\nabla \Delta d\|_{L^2}^{\frac{1}{2}}.$$

To estimate  $I_4$ , notice that the condition  $|d| = 1$ ,

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} [\Delta(|\nabla d|^2) d \cdot \Delta d + 2 \nabla(|\nabla d|^2) \nabla d \cdot \Delta d + |\nabla d|^2 |\Delta d|^2] dx \\
&= \int_{\mathbb{R}^3} [-\nabla(|\nabla d|^2) \nabla d \cdot \Delta d - \nabla(|\nabla d|^2) d \cdot \nabla \Delta d + 2 \nabla(|\nabla d|^2) \nabla d \cdot \Delta d + |\nabla d|^2 |\Delta d|^2] dx \\
&\leq \int_{\mathbb{R}^3} [6 |\nabla d|^2 |\nabla^2 d| |\Delta d| + 2 |\nabla d| |\nabla^2 d| |\nabla \Delta d| + |\nabla d|^2 |\Delta d|^2] dx \\
&\leq C(\|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^\infty} \|\nabla^2 d\|_{L^2} \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2).
\end{aligned}$$

By the standard calculation, it is easy to obtain the equality  $\|\nabla^2 d\|_{L^2} = \|\Delta d\|_{L^2}$ . Hence,

$$\begin{aligned}
I_4 &\leq C(\|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2 + \|\nabla d\|_{L^\infty} \|\nabla^2 d\|_{L^2} \|\nabla \Delta d\|_{L^2}) \\
&\leq \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2 \\
&\leq \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 + C(1 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2 \ln(e + \|\Lambda^3 \nabla d\|_{L^2})) \|\Delta d\|_{L^2}^2 \\
&\leq \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 + C \|\Delta d\|_{L^2}^2 + \\
&\quad C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} [\ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2})]^{\frac{3}{2}} (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2). \quad (2.13)
\end{aligned}$$

Inserting estimates (2.12) and (2.13) into (2.11), it follows that

$$\begin{aligned}
\frac{d}{dt} \|\Delta d(\cdot, t)\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 &\leq \frac{1}{2} \|\nabla \omega\|_{L^2}^2 + C(\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
&\quad + C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} [\ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2})]^{\frac{3}{2}} (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2). \quad (2.14)
\end{aligned}$$

Due to (2.3), one concludes that for any small constant  $\varepsilon > 0$ , there exists  $T = T(\varepsilon) \in (0, T_*)$  such that

$$\int_T^{T_*} \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2)}} dt \leq \varepsilon.$$

For any  $T < t \leq T_*$ , we let

$$y(t) := \sup_{T < \tau \leq t} (\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2). \quad (2.15)$$

Then, by abbreviatingly denoting  $C\{\|\omega(\cdot, T(\varepsilon))\|_{L^2}^2 + \|\Delta d(\cdot, T(\varepsilon))\|_{L^2}^2\}$  as  $C(\varepsilon)$ . Putting (2.10), (2.14) and (2.15) together, we find that

$$\begin{aligned} \frac{d}{dt}(\|\omega(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2) + \|\nabla \omega\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 &\leq C(\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ &+ C \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2)}} [\ln(e + y(t))]^{\frac{3}{2}} (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2). \end{aligned} \quad (2.16)$$

By Gronwall's inequality to (2.16) in the interval  $[T, t]$ , we have

$$\begin{aligned} &\|\omega(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2 \\ &\leq (\|\omega(\cdot, T)\|_{L^2}^2 + \|\Delta d(\cdot, T)\|_{L^2}^2) \exp \left\{ \int_T^t C d\tau + C \int_T^t \frac{\|\omega\|_{\dot{B}_{\infty,\infty}^0}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}_{\infty,\infty}^0}^2 + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2)}} d\tau \ln^{\frac{3}{2}}(e + y(t)) \right\} \\ &\leq (\|\omega(\cdot, T)\|_{L^2}^2 + \|\Delta d(\cdot, T)\|_{L^2}^2) \exp \left\{ C[(T_* - T) + \varepsilon \ln^{\frac{3}{2}}(e + y(t))] \right\} \\ &\leq C(\varepsilon) \exp(C\varepsilon(1 + \ln^{\frac{3}{2}}(e + y(t)))) \leq C(\varepsilon)(e + y(t))^{\frac{3C\varepsilon}{2}}, \end{aligned} \quad (2.17)$$

where  $C$  is the positive constant whose value is independent of either  $\varepsilon$  or  $T$ , and  $C(\varepsilon)$  is a bounded positive constant depending on  $\varepsilon$  which may change from line to line.

Next, we will derive the estimate of  $y(t)$  defined by (2.15). To this end, we need to introduce the following commutator and product estimates (see [10]):

$$\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{\alpha-1}g\|_{L^{q_1}} + \|\Lambda^\alpha f\|_{L^{p_2}} \|g\|_{L^{q_2}}); \quad (2.18)$$

$$\|\Lambda^\alpha(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|\Lambda^\alpha g\|_{L^{q_2}} + \|\Lambda^\alpha f\|_{L^{p_2}} \|g\|_{L^{q_2}}) \quad (2.19)$$

with  $\alpha > 0$ ,  $1 < p, p_1, p_2, q_1, q_2 < \infty$  and  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ .

Applying  $\Lambda^3$  on (1.1), multiplying  $\Lambda^3 u$  and integrating over  $\mathbb{R}^3$ , one obtains

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^3 u(\cdot, t)\|_{L^2}^2 + \|\Lambda^4 u(\cdot, t)\|_{L^2}^2 = - \int_{\mathbb{R}^3} \Lambda^3(u \cdot \nabla u) \cdot \Lambda^3 u dx - \int_{\mathbb{R}^3} \Lambda^3(\Delta d \cdot \nabla d) \cdot \Lambda^3 u dx := I_5 + I_6, \quad (2.20)$$

where we have used the fact that  $\operatorname{div} u = 0$  implies  $\int_{\mathbb{R}^3} \Lambda^3 \nabla(\frac{|\nabla d|^2}{2}) \cdot \Lambda^3 u dx = 0$ . Applying (2.18), it follows that

$$I_5 = \int_{\mathbb{R}^3} [\Lambda^3(u \cdot \nabla u) - u \cdot \nabla \Lambda^3 u] \cdot \Lambda^3 u dx$$



$$\begin{aligned}
&\leq C\|\Lambda^3(u \cdot \nabla u) - u \cdot \nabla \Lambda^3 u\|_{L^{\frac{3}{2}}} \|\Lambda^3 u\|_{L^3} \\
&\leq C\|\nabla u\|_{L^3} \|\Lambda^3 u\|_{L^3}^2 \leq C\|\nabla u\|_{L^2}^{\frac{13}{12}} \|\Lambda^3 u\|_{L^2}^{\frac{1}{4}} \|\Lambda^4 u\|_{L^2}^{\frac{5}{3}} \\
&\leq \frac{1}{4} \|\Lambda^4 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{13}{2}} \|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} \\
&\leq \frac{1}{4} \|\Lambda^4 u\|_{L^2}^2 + C_0 \|\omega\|_{L^2}^{\frac{13}{2}} \|\Lambda^3 u\|_{L^2}^{\frac{3}{2}}.
\end{aligned} \tag{2.21}$$

Here we have used the following Gagliardo-Nirenberg inequalities:

$$\|\nabla u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{\frac{3}{4}} \|\Lambda^3 u\|_{L^2}^{\frac{1}{4}} \text{ and } \|\Lambda^3 u\|_{L^3} \leq C\|\nabla u\|_{L^2}^{\frac{1}{6}} \|\Lambda^4 u\|_{L^2}^{\frac{5}{6}}.$$

For  $I_6$ , applying the Hölder's inequality and the Leibniz's rule, we have

$$\begin{aligned}
I_6 &= \int_{\mathbb{R}^3} \Lambda^2(\Delta d \cdot \nabla d) \cdot \Lambda^4 u dx \\
&\leq \frac{1}{4} \|\nabla^4 u\|_{L^2}^2 + C \int_{\mathbb{R}^3} |\Lambda^2(\Delta d \cdot \nabla d)|^2 dx \\
&\leq \frac{1}{4} \|\nabla^4 u\|_{L^2}^2 + C \int_{\mathbb{R}^3} (|\Lambda^4 d|^2 |\nabla d|^2 + |\Lambda^2 d|^2 |\Lambda^3 d|^2) dx \\
&\leq \frac{1}{4} \|\nabla^4 u\|_{L^2}^2 + C(\|\nabla d\|_{L^6}^2 \|\Lambda^4 d\|_{L^3}^2 + \|\Lambda^2 d\|_{L^4}^2 \|\Lambda^3 d\|_{L^4}^2) \\
&\leq \frac{1}{4} \|\nabla^4 u\|_{L^2}^2 + C(\|\Delta d\|_{L^2}^{\frac{7}{3}} \|\Lambda^5 d\|_{L^2}^{\frac{5}{3}} + \|\Delta d\|_{L^2}^{\frac{19}{6}} \|\Lambda^5 d\|_{L^2}^{\frac{5}{6}}) \\
&\leq \frac{1}{4} \|\nabla^4 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^5 d\|_{L^2}^2 + C_0(\|\Delta d\|_{L^2}^{14} + \|\Delta d\|_{L^2}^{\frac{38}{7}}).
\end{aligned} \tag{2.22}$$

Here we have used the following Gagliardo-Nirenberg inequalities:

$$\begin{aligned}
\|\Lambda^4 d\|_{L^3} &\leq C\|\Delta d\|_{L^2}^{\frac{1}{6}} \|\Lambda^5 d\|_{L^2}^{\frac{5}{6}}; \\
\|\Lambda^2 d\|_{L^4} &\leq C\|\Delta d\|_{L^2}^{\frac{3}{4}} \|\Lambda^5 d\|_{L^2}^{\frac{1}{4}}; \\
\|\Lambda^3 d\|_{L^4} &\leq C\|\Delta d\|_{L^2}^{\frac{5}{6}} \|\Lambda^5 d\|_{L^2}^{\frac{1}{6}}.
\end{aligned}$$

Inserting (2.21) and (2.22) into (2.20), one gets

$$\begin{aligned}
\frac{d}{dt} \|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 u\|_{L^2}^2 &\leq \frac{1}{4} \|\Lambda^5 d\|_{L^2}^2 + C(\|\omega\|_{L^2}^{\frac{13}{2}} \|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + \|\Delta d\|_{L^2}^{14} + \|\Delta d\|_{L^2}^{\frac{38}{7}}) \\
&\leq \frac{1}{4} \|\Lambda^5 d\|_{L^2}^2 + C_0 C(\varepsilon)(1 + y(t))^{\frac{39C\varepsilon}{8}} \|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + C_0 C(\varepsilon)(1 + y(t))^{\frac{21C\varepsilon}{2}}.
\end{aligned} \tag{2.23}$$

Taking  $\Lambda^4$  on (1.2), multiplying  $\Lambda^4 d$  and integrating over  $\mathbb{R}^3$ , one obtains

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^4 d\|_{L^2}^2 + \|\Lambda^5 d\|_{L^2}^2 = - \int_{\mathbb{R}^3} \Lambda^4(u \cdot \nabla d) \cdot \Lambda^4 d dx + \int_{\mathbb{R}^3} \Lambda^4(|\nabla d|^2 d) \cdot \Lambda^4 d dx := I_7 + I_8. \tag{2.24}$$

Similar as the estimate of  $I_5$ , we have

$$\begin{aligned}
I_7 &= - \int_{\mathbb{R}^3} [\Lambda^4(u \cdot \nabla d) - u \cdot \nabla \Lambda^4 d] \cdot \Lambda^4 d dx \\
&\leq C\|\Lambda^4(u \cdot \nabla d) - u \cdot \nabla \Lambda^4 d\|_{L^{\frac{3}{2}}} \|\Lambda^4 d\|_{L^3}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|\nabla d\|_{L^6}\|\Lambda^4 u\|_{L^2}\|\Lambda^4 d\|_{L^3} + C\|\nabla u\|_{L^6}\|\Lambda^4 d\|_{L^2}\|\Lambda^4 d\|_{L^3} \\
&\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2\|\Lambda^4 d\|_{L^3}^2 + \|\Delta u\|_{L^2}\|\Lambda^4 d\|_{L^2}\|\Lambda^4 d\|_{L^3} \\
&\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2\|\Delta d\|_{L^2}^{\frac{1}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{5}{3}} + \|\nabla \omega\|_{L^2}\|\Delta d\|_{L^2}^{\frac{1}{2}}\|\Lambda^5 d\|_{L^2}^{\frac{3}{2}} \\
&\leq \frac{1}{4}\|\Lambda^4 u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^5 d\|_{L^2}^2 + C_0(\|\Delta d\|_{L^2}^{14} + \|\nabla \omega\|_{L^2}^{14} + \|\Delta d\|_{L^2}^{\frac{14}{5}}), \tag{2.25}
\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality:

$$\|\Lambda^4 d\|_{L^2} \leq C\|\Delta d\|_{L^2}^{\frac{1}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{2}{3}} \text{ and } \|\Lambda^4 d\|_{L^3} \leq C\|\Delta d\|_{L^2}^{\frac{1}{6}}\|\Lambda^5 d\|_{L^2}^{\frac{5}{6}}.$$

To estimate  $I_8$ , by using the Leibniz's rule, the fact  $|d| = 1$ , the Hölder's inequality and the Young inequality, one obtains

$$\begin{aligned}
I_8 &= \int_{\mathbb{R}^3} \Lambda^4(|\nabla d|^2 d) \cdot \Lambda^4 d dx = - \int_{\mathbb{R}^3} \Lambda^3(|\nabla d|^2 d) \cdot \Lambda^5 d dx \\
&= - \int_{\mathbb{R}^3} [\Lambda^3(|\nabla d|^2) d \cdot \Lambda^5 d + 3\Lambda^2(|\nabla d|^2) \Lambda d \cdot \Lambda^5 d + 3\Lambda(|\nabla d|^2) \Lambda^2 d \cdot \Lambda^5 d + |\nabla d|^2 \Lambda^3 d \cdot \Lambda^5 d] dx \\
&\leq C\|\Lambda^5 d\|_{L^2}(\|\nabla d\|_{L^6}\|\Lambda^4 d\|_{L^3} + \|\Lambda^2 d\|_{L^4}\|\Lambda^3 d\|_{L^4} + \|\nabla d\|_{L^6}^2\|\Lambda^3 d\|_{L^6} + \|\nabla d\|_{L^6}\|\Lambda^2 d\|_{L^6}^2) \\
&\leq C\|\Lambda^5 d\|_{L^2}(\|\Delta d\|_{L^2}^{\frac{7}{6}}\|\Lambda^5 d\|_{L^2}^{\frac{5}{6}} + \|\Delta d\|_{L^2}^{\frac{7}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{2}{3}}) \\
&\leq \frac{1}{4}\|\Lambda^5 d\|_{L^2}^2 + C_0\|\Delta d\|_{L^2}^{14}. \tag{2.26}
\end{aligned}$$

Here we have used the following Gagliardo-Nirenberg inequalities:

$$\begin{aligned}
\|\Lambda^4 d\|_{L^3} &\leq C\|\Delta d\|_{L^2}^{\frac{1}{6}}\|\Lambda^5 d\|_{L^2}^{\frac{5}{6}}; \\
\|\Lambda^2 d\|_{L^4} &\leq C\|\Delta d\|_{L^2}^{\frac{3}{4}}\|\Lambda^5 d\|_{L^2}^{\frac{1}{4}}; \\
\|\Lambda^3 d\|_{L^6} &\leq C\|\Delta d\|_{L^2}^{\frac{1}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{2}{3}}; \\
\|\Lambda^2 d\|_{L^6} &\leq C\|\Delta d\|_{L^2}^{\frac{2}{3}}\|\Lambda^5 d\|_{L^2}^{\frac{1}{3}}.
\end{aligned}$$

Inserting (2.25) and (2.26) into (2.24), one gets

$$\begin{aligned}
\frac{d}{dt}\|\Lambda^4 d\|_{L^2}^2 + \|\Lambda^5 d\|_{L^2}^2 &\leq \frac{1}{2}\|\Lambda^4 d\|_{L^2}^2 + C_0(\|\Delta d\|_{L^2}^{14} + \|\nabla \omega\|_{L^2}^{14} + \|\Delta d\|_{L^2}^{\frac{14}{5}}) \\
&\leq \frac{1}{2}\|\Lambda^4 d\|_{L^2}^2 + C_0 C(\varepsilon)(1 + y(t))^{\frac{21C\varepsilon}{2}}. \tag{2.27}
\end{aligned}$$

Combining (2.23) and (2.27) together, one obtains

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2) + \frac{1}{2}(\|\Lambda^4 u\|_{L^2}^2 + \|\Lambda^5 d\|_{L^2}^2) \\
&\leq C_0 C(\varepsilon)(e + y(t))^{\frac{39C\varepsilon}{8}}\|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + C_0 C(\varepsilon)(e + y(t))^{\frac{21C\varepsilon}{2}} \\
&\leq C_0 C(\varepsilon)(e + y(t))^{\frac{21C\varepsilon}{2}}(1 + \|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2)^{\frac{3}{4}} \\
&\leq C_0 C(\varepsilon)(e + y(t))^{\frac{3}{4} + \frac{21C\varepsilon}{2}}.
\end{aligned}$$

Hence

$$\frac{d}{dt}y(t) \leq C_0 C(\varepsilon)(e + y(t))^{\frac{3}{4} + \frac{21C\varepsilon}{2}}. \tag{2.28}$$

By selecting  $\varepsilon$  sufficiently small such that  $\frac{3}{4} + \frac{21C\varepsilon}{2} < 1$ , then applying Gronwall's inequality to the above inequality (2.28), we get the boundness of  $y(t)$  on  $[T, T_*]$ , i.e., the estimate (2.4) is proved under the condition (1.6). This completes the proof of Theorem 1.1 under the condition (1.6).

*Case II:* Next, we prove Theorem 1.1 under the condition (1.7). Multiplying (1.1) by  $-\Delta u$ , integrating over  $\mathbb{R}^3$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(\cdot, t)\|_{L^2}^2 + \|\Delta u(\cdot, t)\|_{L^2}^2 &= - \int_{\mathbb{R}^3} (\nabla u \cdot \nabla) u \cdot \nabla u dx + \int_{\mathbb{R}^3} (\nabla d \cdot \Delta d) \cdot \Delta u dx \\ &= - \int_{\mathbb{R}^3} (\nabla u \cdot \nabla) u \cdot \nabla u dx + \sum_{i,k=1}^3 \int_{\mathbb{R}^3} \partial_i d_k \Delta d_k \Delta u_i dx. \end{aligned} \quad (2.29)$$

Applying  $\Delta$  on (1.2), multiplying it with  $\Delta d$ , and using (1.3), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta d(\cdot, t)\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 &= - \int_{\mathbb{R}^3} \Delta(u \cdot \nabla d) \cdot \Delta d dx + \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d) \cdot \Delta d dx \\ &= - \int_{\mathbb{R}^3} (\Delta u \cdot \nabla d) \cdot \Delta d dx - 2 \int_{\mathbb{R}^3} (\nabla u \cdot \nabla) \nabla d \cdot \Delta d dx + \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d) \cdot \Delta d dx \\ &= - \sum_{i,k=1}^3 \int_{\mathbb{R}^3} \partial_i d_k \Delta d_k \Delta u_i dx - 2 \int_{\mathbb{R}^3} (\nabla u \cdot \nabla) \nabla d \cdot \Delta d dx + \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d) \cdot \Delta d dx. \end{aligned} \quad (2.30)$$

Combining (2.29) and (2.30) together, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) \\ &= - \int_{\mathbb{R}^3} (\nabla u \cdot \nabla) u \cdot \nabla u dx - 2 \int_{\mathbb{R}^3} (\nabla u \cdot \nabla) \nabla d \cdot \Delta d dx + \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d) \Delta d dx \\ &=: J_1 + J_2 + J_3. \end{aligned} \quad (2.31)$$

For  $J_1$  and  $J_2$ , by the Hölder's inequality and (2.1), we have

$$\begin{aligned} J_1 &\leq \int_{\mathbb{R}^3} |\nabla u| |\nabla u| |\nabla u| dx \leq C \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^2} \\ &\leq C \|\Delta u\|_{L^2} \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}} \|\nabla u\|_{L^2} \\ &\leq \frac{1}{2} \|\Delta u\|_{L^2}^2 + C \|\nabla u\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|\nabla u\|_{L^2}^2; \end{aligned} \quad (2.32)$$

$$\begin{aligned} J_2 &\leq C \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^4} \|\Delta d\|_{L^4} \leq C \|\nabla u\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla \Delta d\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2, \end{aligned} \quad (2.33)$$

where we have used the Gagliardo-Nirenberg inequalities

$$\|\nabla^2 d\|_{L^4} \leq C \|\nabla d\|_{L^\infty}^{\frac{1}{2}} \|\nabla \Delta d\|_{L^2}^{\frac{1}{2}} \text{ and } \|\Delta d\|_{L^4} \leq C \|\nabla d\|_{L^\infty}^{\frac{1}{2}} \|\Delta d\|_{L^2}^{\frac{1}{2}}.$$

For  $J_3$ , similar as the estimate of  $I_4$ , we have

$$J_3 \leq \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2. \quad (2.34)$$

Substituting the above estimates (2.32)–(2.34) into (2.31), we obtain

$$\begin{aligned}
& \frac{d}{dt} (\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) \\
& \leq C \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
& \leq C (\|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{L^\infty}^2) (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
& \leq C (1 + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2 \ln(e + \|\Lambda^3 \nabla d\|_{L^2})) (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
& \leq C \left( 1 + \frac{\|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}} \right) \sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})} \\
& \quad \times \ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2}) (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
& \leq C \left( 1 + \frac{\|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}} \right) \sqrt{1 + \ln(e + \|u\|_{\dot{B}_{\infty, \infty}^0} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})} \\
& \quad \times \ln(e + \|\Lambda^3 u\|_{L^2} + \|\Lambda^4 d\|_{L^2}) (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
& \leq C \left( 1 + \frac{\|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}} \right) (1 + \ln(e + y(t)))^{\frac{3}{2}} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \quad (2.35)
\end{aligned}$$

where  $y(t)$  is defined as (2.15). Here we have used the Sobolev imbedding (2.8) again. Similarly, assume that the condition (1.7) is not true, one concludes that for any small constant  $\varepsilon > 0$ , there exists  $T = T(\varepsilon) \in (0, T_*)$  such that

$$\int_T^{T_*} \frac{\|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}} dt \leq \varepsilon. \quad (2.36)$$

Applying Gronwall's inequality on (2.36) for the interval  $[T, t]$ , one obtains

$$\begin{aligned}
& \|\nabla u(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2 \\
& \leq (\|\nabla u(\cdot, T)\|_{L^2}^2 + \|\Delta d(\cdot, T)\|_{L^2}^2) \exp \left\{ \int_T^t C d\tau + C \int_T^t \frac{\|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}}^2 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla u\|_{\dot{B}_{\infty, \infty}^{-1}} + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}} d\tau \ln^{\frac{3}{2}}(e + y(t)) \right\} \\
& \leq (\|\nabla u(\cdot, T)\|_{L^2}^2 + \|\Delta d(\cdot, T)\|_{L^2}^2) \exp \left\{ C[(T_* - T) + \varepsilon \ln^{\frac{3}{2}}(e + y(t))] \right\} \\
& \leq C(\varepsilon) \exp(C\varepsilon(1 + \ln^{\frac{3}{2}}(e + y(t)))) \leq C(\varepsilon)(e + y(t))^{\frac{3C\varepsilon}{2}}. \quad (2.37)
\end{aligned}$$

Similar as the proofs in *Case I*, by using the above estimate (2.37), we find that (2.28) still remains valid. Thus, we deduce that the estimate (2.4) still holds provided that the  $\varepsilon$  in (2.36) sufficiently small, i.e., (2.4) holds under the condition (1.7). This completes the proof of Theorem 1.1 under (1.7).  $\square$

### 3 Proof of Theorem 1.3

Similar as the proof of Theorem 1.1, we prove Theorem 1.3 by contradiction. Assume that (1.8) is not true, then there exists  $0 < M < \infty$  such that

$$\int_0^{T_*} \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt \leq M. \quad (3.1)$$

In what follows, we will give some a priori estimates to show that

$$\lim_{t \rightarrow T_*^-} (\|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \|\nabla^3 d(t, \cdot)\|_{L^2}^2) \leq C \quad (3.2)$$

for some positive constant  $C$  depends only on  $u_0, d_0, T_*$  and  $M$ .

We first taking  $\nabla \times$  on (1.1), it follows from dimension  $n = 2$  that

$$\omega_t - \Delta \omega + u \cdot \nabla \omega = -\nabla \times (\Delta d \cdot \nabla d).$$

Multiplying above equality with  $\omega$  and integrating over  $\mathbb{R}^2$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega(\cdot, t)\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 &= - \int_{\mathbb{R}^2} \nabla \times (\Delta d \cdot \nabla d) \cdot \omega dx \\ &= \int_{\mathbb{R}^2} (\Delta d \cdot \nabla d) \cdot \nabla \times \omega dx \leq C \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} \|\nabla \omega\|_{L^2} \\ &\leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2. \end{aligned} \quad (3.3)$$

Similar as the estimate of  $\Delta d$  on the *Case I* in section 2, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta d(\cdot, t)\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 &= - \int_{\mathbb{R}^2} \Delta(u \cdot \nabla d) \cdot \Delta d dx + \int_{\mathbb{R}^2} \Delta(|\nabla d|^2 d) \cdot \Delta d dx \\ &\leq \int_{\mathbb{R}^2} (|\Delta u| |\nabla d| |\Delta d| + 2 |\nabla u| |\nabla^2 d| |\Delta d| + \Delta(|\nabla d|^2) d \cdot \Delta d + 2 \nabla(|\nabla d|^2) \nabla d \cdot \Delta d + |\nabla d|^2 |\Delta d|^2) dx \\ &\leq C (\|\Delta u\|_{L^2} \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla^2 d\|_{L^4} \|\Delta d\|_{L^4} \\ &\quad + \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^\infty} \|\nabla^2 d\|_{L^2} \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2) \\ &\leq C (\|\nabla \omega\|_{L^2} \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^2} + \|\omega\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^\infty}^2 \|\Delta d\|_{L^2}^2) \\ &\leq \frac{1}{4} \|\nabla \omega\|_{L^2}^2 + \frac{1}{2} \|\nabla \Delta d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \end{aligned} \quad (3.4)$$

where we have used the facts  $|d| = 1$ ,  $\|\Delta u\|_{L^2} \leq C \|\nabla \omega\|_{L^2}$ ,  $\|\nabla u\|_{L^2} \leq C \|\omega\|_{L^2}$  and  $\|\nabla^2 d\|_{L^2} = \|\Delta d\|_{L^2}$ , and the Gagliardo-Nirenberg inequalities in  $\mathbb{R}^2$ :

$$\|\nabla^2 d\|_{L^4} \leq C \|\nabla d\|_{L^\infty}^{\frac{1}{2}} \|\nabla \Delta d\|_{L^2}^{\frac{1}{2}} \text{ and } \|\Delta d\|_{L^4} \leq C \|\nabla d\|_{L^\infty}^{\frac{1}{2}} \|\Delta d\|_{L^2}^{\frac{1}{2}}.$$

Notice that due to (3.1), one concludes that for any small constant  $\varepsilon > 0$ , there exists  $T = T(\varepsilon) \in (0, T_*)$  such that

$$\int_T^{T_*} \frac{\|\nabla d\|_{\dot{B}_{\infty,\infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty,\infty}^0})}} dt \leq \varepsilon.$$

For any  $T < t \leq T_*$ , let

$$z(t) := \sup_{T < \tau \leq t} (\|\Lambda^2 u\|_{L^2}^2 + \|\Lambda^3 d\|_{L^2}^2). \quad (3.5)$$

Combining (3.3) and (3.4) together, and using the inequality (2.2) with  $s = 3$ , it follows that

$$\begin{aligned} & \frac{d}{dt} (\|\omega(\cdot, t)\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + (\|\nabla \omega\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) \leq C \|\nabla d\|_{L^\infty}^2 (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ & \leq C(1 + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2 \ln(e + \|\Lambda^2 \nabla d\|_{L^2})) (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ & \leq C(1 + \frac{\|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}}) \sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})} \ln(e + \|\Lambda^3 d\|_{L^2}) (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ & \leq C(1 + \frac{\|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}}) (\ln(e + z(t)))^{\frac{3}{2}} (\|\omega\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \end{aligned} \quad (3.6)$$

where we have used the Sobolev imbedding

$$H^2(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2) \subseteq \dot{B}_{\infty, \infty}^0(\mathbb{R}^2).$$

By Gronwall's inequality to (3.6) in the interval  $[T, t]$ , we have

$$\begin{aligned} & \|\omega(\cdot, t)\|_{L^2}^2 + \|\Delta d(\cdot, t)\|_{L^2}^2 \\ & \leq (\|\omega(\cdot, T)\|_{L^2}^2 + \|\Delta d(\cdot, T)\|_{L^2}^2) \exp \left\{ \int_T^t C d\tau + C \int_T^t \frac{\|\nabla d\|_{\dot{B}_{\infty, \infty}^0}^2}{\sqrt{1 + \ln(e + \|\nabla d\|_{\dot{B}_{\infty, \infty}^0})}} d\tau (\ln(e + y(t)))^{\frac{3}{2}} \right\} \\ & \leq (\|\omega(\cdot, T)\|_{L^2}^2 + \|\Delta d(\cdot, T)\|_{L^2}^2) \exp \left\{ C[(T_* - T) + \varepsilon \ln^{\frac{3}{2}}(e + z(t))] \right\} \\ & \leq C(\varepsilon) \exp(C\varepsilon(1 + \ln^{\frac{3}{2}}(e + z(t)))) \leq C(\varepsilon)(e + z(t))^{\frac{3C\varepsilon}{2}}. \end{aligned} \quad (3.7)$$

On the other hand, by Lin, Lin and Wang [17], we known that when the dimension  $n = 2$ , there holds the energy equality

$$\|u(\cdot, t)\|_{L^2}^2 + \|\nabla d(\cdot, t)\|_{L^2}^2 + \int_0^{T_*} \int_{\mathbb{R}^2} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|) dx dt = \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2. \quad (3.8)$$

Now, we will estimate  $z(t)$  defined by (3.5). Let us recall the following useful Gagliardo-Nirenberg inequalities in  $\mathbb{R}^2$ :

$$\begin{aligned} \|\nabla u\|_{L^3} & \leq C \|\nabla u\|_{L^2}^{\frac{5}{6}} \|\Lambda^3 u\|_{L^2}^{\frac{1}{6}}; & \|\nabla u\|_{L^6} & \leq C \|u\|_{L^2}^{\frac{4}{9}} \|\Lambda^3 u\|_{L^2}^{\frac{5}{9}}; \\ \|\Lambda^2 u\|_{L^3} & \leq C \|\nabla u\|_{L^2}^{\frac{1}{3}} \|\Lambda^3 u\|_{L^2}^{\frac{2}{3}}; & \|\nabla d\|_{L^6} & \leq C \|\nabla d\|_{L^2}^{\frac{1}{3}} \|\Delta d\|_{L^2}^{\frac{2}{3}}; \\ \|\Lambda^2 d\|_{L^4} & \leq C \|\Delta d\|_{L^2}^{\frac{3}{4}} \|\Lambda^4 d\|_{L^2}^{\frac{1}{4}}; & \|\Lambda^3 d\|_{L^3} & \leq C \|\Delta d\|_{L^2}^{\frac{1}{3}} \|\Lambda^4 d\|_{L^2}^{\frac{2}{3}}. \end{aligned} \quad (3.9)$$

Applying  $\Lambda^2$  on (1.1), multiplying  $\Lambda^2 u$  and integrating over  $\mathbb{R}^3$ , and using (2.18), the Hölder's inequality, (3.9) and the Young inequalities, one obtains

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^2 u(\cdot, t)\|_{L^2}^2 + \|\Lambda^3 u\|_{L^2}^2 = - \int_{\mathbb{R}^2} \Lambda^2(u \cdot \nabla u) \cdot \Lambda^2 u dx - \int_{\mathbb{R}^2} \Lambda^2(\Delta d \cdot \nabla d) \cdot \Lambda^2 u dx$$

$$\begin{aligned}
&= - \int_{\mathbb{R}^2} [\Lambda^2(u \cdot \nabla u) - u \cdot \nabla \Lambda^2 u] \cdot \Lambda^2 u dx + \int_{\mathbb{R}^2} \Lambda(\Delta d \cdot \nabla d) \cdot \Lambda^3 u dx \\
&\leq C \|[\Lambda^2(u \cdot \nabla u) - u \cdot \nabla \Lambda^2 u]\|_{L^{\frac{3}{2}}} \|\Lambda^2 u\|_{L^3} + C(\|\nabla d\|_{L^6} \|\Lambda^3 d\|_{L^3} \|\Lambda^3 u\|_{L^2} + \|\Lambda^3 u\|_{L^2} \|\Lambda^2 d\|_{L^4}^2) \\
&\leq C(\|\Lambda^2 u\|_{L^3}^2 \|\nabla u\|_{L^3} + \|\nabla d\|_{L^6} \|\Lambda^3 d\|_{L^3} \|\Lambda^3 u\|_{L^2} + \|\Lambda^3 u\|_{L^2} \|\Lambda^2 d\|_{L^4}^2) \\
&\leq C(\|\nabla u\|_{L^2}^{\frac{3}{2}} \|\Lambda^3 u\|_{L^2}^{\frac{3}{2}} + \|\Lambda^3 u\|_{L^2} \|\nabla d\|_{L^2}^{\frac{1}{3}} \|\Delta d\|_{L^2} \|\Lambda^4 d\|_{L^2}^{\frac{2}{3}} + \|\Lambda^3 u\|_{L^2} \|\Delta d\|_{L^2}^{\frac{3}{2}} \|\Lambda^4 d\|_{L^2}^{\frac{1}{2}}) \\
&\leq \frac{1}{4} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^4 d\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^6 + \|\nabla d\|_{L^2}^2 \|\Delta d\|_{L^2}^6 + \|\Delta d\|_{L^2}^6) \\
&\leq \frac{1}{4} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^4 d\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^6 + \|\Delta d\|_{L^2}^6) \\
&\leq \frac{1}{4} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^4 d\|_{L^2}^2 + C(1 + \|\omega\|_{L^2}^6 + \|\Delta d\|_{L^2}^6), \tag{3.10}
\end{aligned}$$

where we have used the energy equality (3.8). Taking  $\Lambda^3$  on (1.2), multiplying  $\Lambda^3 d$ , integrating over  $\mathbb{R}^2$ , and using (2.18), the Hölder's inequality, (3.9) and the Young inequalities, one obtains

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\Lambda^3 d(\cdot, t)\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2 = - \int_{\mathbb{R}^2} \Lambda^3(u \cdot \nabla d) \cdot \Lambda^3 d dx + \int_{\mathbb{R}^2} \Lambda^3(|\nabla d|^2 d) \cdot \Lambda^3 d dx \\
&= - \int_{\mathbb{R}^2} [\Lambda^3(u \cdot \nabla d) - u \cdot \nabla \Lambda^3 d] \cdot \Lambda^3 d dx - \int_{\mathbb{R}^2} \Lambda^2(|\nabla d|^2 d) \cdot \Lambda^4 d dx \\
&= - \int_{\mathbb{R}^2} [\Lambda^3(u \cdot \nabla d) - u \cdot \nabla \Lambda^3 d] \cdot \Lambda^3 d dx - \int_{\mathbb{R}^2} [\Lambda^2(|\nabla d|^2 d) + 2\Lambda(|\nabla d|^2) \Lambda d + |\nabla d|^2 \Lambda^2 d] \cdot \Lambda^4 d dx \\
&\leq C \|[\Lambda^3(u \cdot \nabla d) - u \cdot \nabla \Lambda^3 d]\|_{L^{\frac{3}{2}}} \|\Lambda^3 d\|_{L^3} + C(\|\Lambda^3 d\|_{L^3} \|\nabla d\|_{L^6} + \|\Lambda^2 d\|_{L^4}^2 \\
&\quad + \|\Lambda d\|_{L^6}^2 \|\Lambda^2 d\|_{L^6} + \|\nabla d\|_{L^6} \|\Lambda^3 d\|_{L^3}) \|\Lambda^4 d\|_{L^2} \\
&\leq C(\|\Lambda^3 d\|_{L^3} \|\nabla d\|_{L^6} \|\Lambda^3 u\|_{L^2} + \|\Lambda^3 d\|_{L^3}^2 \|\nabla u\|_{L^6} + \|\Lambda^3 d\|_{L^3} \|\nabla d\|_{L^6} \|\Lambda^4 d\|_{L^2} \\
&\quad + \|\Lambda^2 d\|_{L^4}^2 \|\Lambda^4 d\|_{L^2} + \|\Lambda d\|_{L^6}^2 \|\Lambda^2 d\|_{L^6} \|\Lambda^4 d\|_{L^2}) \\
&\leq \frac{1}{8} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{8} \|\Lambda^4 d\|_{L^2}^2 + C(\|\nabla d\|_{L^6}^2 \|\Lambda^3 d\|_{L^3}^2 + \|\nabla u\|_{L^6} \|\Lambda^3 d\|_{L^3}^2 + \|\Lambda^2 d\|_{L^4}^4 + \|\Lambda d\|_{L^6}^4 \|\Lambda^2 d\|_{L^6}^2) \\
&\leq \frac{1}{8} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{8} \|\Lambda^4 d\|_{L^2}^2 + C(\|\nabla d\|_{L^2}^{\frac{2}{3}} \|\Delta d\|_{L^2}^2 \|\Lambda^4 d\|_{L^2}^{\frac{4}{3}} + \|u\|_{L^2}^{\frac{4}{9}} \|\Lambda^3 u\|_{L^2}^{\frac{5}{9}} \|\Delta d\|_{L^2}^{\frac{2}{3}} \|\Lambda^4 d\|_{L^2}^{\frac{4}{3}} \\
&\quad + \|\Delta d\|_{L^2}^3 \|\Lambda^4 d\|_{L^2} + \|\nabla d\|_{L^2}^{\frac{4}{3}} \|\Delta d\|_{L^2}^4 \|\Lambda^4 d\|_{L^2}^{\frac{2}{3}}) \\
&\leq \frac{1}{4} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^4 d\|_{L^2}^2 + C(\|\nabla d\|_{L^2}^2 \|\Delta d\|_{L^2}^6 + \|u\|_{L^2}^8 \|\Delta d\|_{L^2}^{12} + \|\Delta d\|_{L^2}^6 + \|\nabla d\|_{L^2}^2 \|\Delta d\|_{L^2}^6) \\
&\leq \frac{1}{4} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^4 d\|_{L^2}^2 + C(\|\Delta d\|_{L^2}^6 + \|\Delta d\|_{L^2}^{12}) \\
&\leq \frac{1}{4} \|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4} \|\Lambda^4 d\|_{L^2}^2 + C(1 + \|\Delta d\|_{L^2}^{12}), \tag{3.11}
\end{aligned}$$

where we have used the equality (3.8). Combining (3.10) and (3.11) together, and using estimate (3.7), we obtain

$$\begin{aligned}
&\frac{d}{dt} (\|\Lambda^2 u(\cdot, t)\|_{L^2}^2 + \|\Lambda^3 d(\cdot, t)\|_{L^2}^2) + (\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2) \\
&\leq C(1 + \|\omega\|_{L^2}^6 + \|\Delta d\|_{L^2}^{12}) \leq C(1 + \|\omega\|_{L^2}^{12} + \|\Delta d\|_{L^2}^{12}) \\
&\leq C(\varepsilon)(1 + z(t))^{9C\varepsilon}. \tag{3.12}
\end{aligned}$$

By selecting  $\varepsilon$  sufficiently small such that  $9C\varepsilon < 1$ , and applying Gronwall's inequality to the above inequality (3.12), we get the boundness of  $z(t)$  on  $[T, T_*]$ , i.e., the estimate (3.2) is proved under the condition (1.8). This completes the proof of Theorem 1.3.  $\square$

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